Abstract— In this paper, we introduce a new concept of additive color Laplacian energy of a graph, \( LE_{ac}(G) \) and compute the additive color Laplacian energy \( LE_{ac}(G) \) of few families of graphs with minimum number of colors. It depends on the underlying graph and colors on its vertices. We establish some bounds for additive color Laplacian energy. We also obtained semi color Laplacian energy of complement of colored graphs of few families of graphs.

Keywords- Additive color Laplacian matrix, Additive color Laplacian eigenvalues, Additive color Laplacian energy.

I. INTRODUCTION

For standard terminology and notation in graph theory[1], the concept of graph energy originates from chemistry to estimate the total -electron energy of a molecule. In chemistry the conjugated hydrocarbons can be represented by a graph called molecular graph. Here every carbon atom is represented by a vertex and every carbon-carbon bond by an edge and hydrogen atoms are ignored. The eigenvalues of the molecular graph represent the energy level of the electron in the molecule. An interesting quantity in Huckel theory is the sum of the energies of all the electrons in a molecule, the so called \(-\)electron energy of a molecule. In spectral graph theory, the eigenvalues of several matrices like adjacency matrix, Laplacian matrix, distance matrix, maximum degree matrix, minimum degree matrix, matrix of a subset \( S \) of \( V \) and color energy of a graph are studied in [5],[3],[2],[4],[11],[21].

A coloring of graph \( G \) is a coloring of its vertices such that no two adjacent vertices receive the same color. The minimum number of colors needed for coloring of a graph \( G \) is called chromatic number and denoted by \((G)\). In particular, we consider the labeling to be a coloring then we have the vertex colored graph. Then entries of the matrix \( AL(G) \) are as follows: If \( c(v_i) \) is the color of \( v_i \), then

\[
\begin{align*}
-1 & \quad \text{if } v_i \text{ and } v_j \text{ are non-adjacent with } c(v_i) = c(v_j), \\
1 & \quad \text{if } v_i \text{ and } v_j \text{ are adjacent with } c(v_i) \neq c(v_j), \\
0 & \quad \text{otherwise}
\end{align*}
\]

The matrix thus obtained is the \( L \)-matrix of the colored graph \( G \) denoted by \( Ac(G) \). The eigenvalues of \( Ac(G) \) are called color eigenvalues. The color energy of a graph denoted by \( Ec(G) \) is the sum of the absolute values of the color eigenvalues of \( G \), i.e.,

\[
Ec(G) = \sum_{i=0}^{n} |\lambda_i|
\]  

(1)

In [1] and [2], Chandrashekar Adiga et.al has studied the energy of the colored graph. They even introduced the compliment of a colored graph.

Definition 1.1 Let \( G = (V; E) \) be a colored graph. Then the complement of colored graph \( G \) denoted by \( G_c \) has same vertex set and same coloring of \( G \) with the following properties:

1. \( v_i \) and \( v_j \) are adjacent in \( G_c \), if \( v_i \) and \( v_j \) are non-adjacent in \( G \) with \( c(v_i) \neq c(v_j) \):
2. $v_i$ and $v_j$ are non-adjacent in $G$, if $\overline{v_i}$ and $v_j$ are non-adjacent in $G$ with $c(\overline{v_i}) = c(v_j)$;
(iii) $v_i$ and $v_j$ are non-adjacent in $G$, if $v_i$ and $v_j$ are adjacent in $G$.

Definition 1.2. $D(G)$ be the diagonal matrix of vertex degrees of the graph $G$. Then
$L(G) = D(G) A(G)$ is the Laplacian matrix of Graph $G$. Let $\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_n$ be the Laplacian eigenvalues of graph $G$. The Laplacian energy of the graph $G$ is defined as

$$LE(G) = \sum_{i=1}^{n} \frac{2m_{\bar{\lambda}_i}}{n}$$

Recently in [18], P. G. Bhat and Sabitha Dsouza introduced Color Laplacian energy of a graph. The below definition gives color Laplacian matrix.

Definition 1.4. If $G$ be a colored graph on $n$ vertices and $m$ edges, the color Laplacian matrix of $G$ is given by $Lc(G) = D(G) - Ac(G)$.

Definition 1.5. For given graph, the matrix $Q = D + A$ is called the signless Laplacian, where $A$ is the adjacency matrix and $D$ is the diagonal matrix of vertex degrees.

This paper is organized as follows. In section 2, additive color Laplacian energy is defined. In section 3, additive color Laplacian spectrum and additive color Laplacian energies are derived for some families of graphs and their complements. At the end we propose some open problems.

Definition 1.6. For given graph, the matrix $Q = D + A$ is called the signless Laplacian, where $A$ is the adjacency matrix and $D$ is the diagonal matrix of vertex degrees.

This paper is organized as follows. In section 2, we establish some general results on Additive color Laplacian eigenvalues and we define Additive color Laplacian energy of a graph. In section 3, lower and upper bounds for Additive color Laplacian energy of a graph are obtained. In section 4 Additive color Laplacian spectrum and Additive color Laplacian energies are derived for some families of graphs and their complements.

II. ADDITIVE COLOR LAPLACIAN ENERGY OF A GRAPH

Let $G$ be a graph on $n$ vertices and $m$ edges. We define the additive color Laplacian matrix of $G$ as $ALc = D(G) + Ac(G)$. The eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$ of $ALc$ are called Additive color Laplacian eigenvalues of the graph $G$.

III. ADDITIVE COLOR LAPLACIAN SPECTRUM, ESTIMATION OF ADDITIVE COLOR LAPLACIAN ENERGY OF SOME FAMILIES OF GRAPHS AND THEIR COLOR COMPLEMENT

In this section, we consider characteristic polynomial, Additive color Laplacian spectrum and Additive color Laplacian energy of colored graphs by giving minimum colors and we denote them as $P(G)$; $\gamma$; $ALspectrum$ and $ALE(G)$ respectively.

Theorem 3.1. If $K_n$ is the complete graph of order $n$, then $ALE(K_n) = 2(n-1)$.

Proof. Let $K_n$ be the complete graph with vertex set $V = v_1; v_2; \ldots; v_n$. Since $(K_n) = n$ we have

$$(n-1) 1 1 1 \ldots 1 1
1 n-1 1 1 \ldots 1 1
1 1 (n-1) 1 \ldots 1 1
1 1 1 1 \ldots 1 1
1 1 1 0 \ldots (n-1) 1
1 1 1 0 \ldots 1 n-1)$$

Additive color Laplacian spectra is $n-2$ ($n-1$ times), $2n-2$ (one time)

Thus $ALE(K_n) = 2(n-1)$.
Remark 3.2. The Additive color Laplacian energy of complete graph is same as the energy of complete graph.

Theorem 3.3. If (K_{n})c is the complement of the colored graph K_{n}, then ALE (K_{n})c = 0.

Proof. Let (K_{n}), be the complement of the colored graph K_{n}. Then A_{c}(G)= (0)_{n×n}
The characteristic polynomial is P ((K_{n})c) = μ^{n}.
Hence, ALE (K_{n})c = 0.

Theorem 3.4. If G is a null graph of order n, then its additive color Laplacian energy ALE (G) = 2(n-1).

Proof. The proof is simple we omit the proof.

Theorem 3.5. If (K_{1}, n-1) the complement of the colored star graph of order n, then the additive color Laplacian Energy is \[
\frac{2n - 4}{n} + 2\sqrt{(n-1)(n-2)}
\]

Proof. Let (K_{1}, n-1), be the complement of the colored star graph. Then
\[
\begin{pmatrix}
n-1 & -1 & -1 & ... & -1 & -1 \\
-1 & 1 & -1 & ... & -1 & -1 \\
-1 & -1 & 1 & ... & -1 & -1 \\
-1 & -1 & -1 & 1 & ... & -1 \\
-1 & -1 & -1 & 0 & ... & 1 \\
-1 & -1 & -1 & 0 & ... & 1 \\
\end{pmatrix}
\]

The additive color Laplacian Energy is \[
\frac{2n - 4}{n} + 2\sqrt{(n-1)(n-2)}
\]

Definition 3.6. The Crown graph S_{n}^{b} for an integer n ≥3 is the graph with vertex set \{u_{1}, u_{2},..., u_{n}, v_{1}, v_{2},...,v_{n}\} and edge set \{u_{i}v_{j} : 1 ≤i, j ≤n; i ≠j\}. S_{n}^{b} is therefore equivalent to the complete bipartite graph K_{n,n} with horizontal edges removed.

Theorem 3.7. If Sn0 is a crown graph of order 2n, then the additive color Laplacian energy is ALE (Sn0) = 4(n -1).

Proof. The proof is similar to the theorem 3.1.

Theorem 3.8: If Knx2 be the Cock tail party graph of order “n”. Then the additive color Laplacian energy is ALE (Knx2) = 6(n -1).

Proof. The proof is similar to the theorem 3.1.

Theorem 3.9: If Kn × Z be the complement of Cock tail party graph of order “n”. Then the additive color Laplacian energy is ALE (Kn × Z) = 2n.

Proof. The proof is similar to the theorem 3.1.

IV. SOME OPEN PROBLEMS

Problem 4.1. Determine the class of graphs whose Additive Color Laplacian energy of a graph is equal to number of vertices.

Problem 4.2. Determine the class of graphs whose Additive Color Laplacian energy of complement colored graph is equal to number of vertices.

Problem 4.3. Determine the class of graphs whose Additive Color Laplacian energy of a graph equal to usual energy.

Problem 4.4. Determine the class of graphs whose Additive Color Laplacian energy and energy of the complement colored graphs are equal.

REFERENCES