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Minimum Hub Energy of a Graph

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Abstract—**Recently Prof. Chandrashekar Adiga et al., have defined the minimum covering energy, of a graph G which depends on its particular minimum cover C. Motivated by this paper, we introduced the concept of minimum hub energy $E_H(G)$ of a graph G and computed minimum hub energies of some standard graphs.**

***Keywords*— minimum Hub set, minimum Hub eigenvalues, minimum Hub energy of a graph.**

I. Introduction

Let G be a graph with n vertices v_1, v_2....v_n and m edges. Let A be the adjacency matrix of the graph. The eigenvalues λ_1, λ_2....λ_n of A, assumed in non-increasing order, are the eigenvalues of the graph G. As A is real symmetric, the eigenvalues of G are real with sum equal to zero. The energy E(G) of G is defined to be the sum of the absolute values of the eigenvalues of G.

The concept of graph energy originates from chemistry to estimate the total π-electron energy of a molecule. In chemistry the conjugated hydrocarbons can be represented by a graph called molecular graph. Here every carbon atom is represented by a vertex and every carbon-carbon bond by an edge and hydrogen atoms are ignored. The eigenvalues of the molecular graph represent the energy level of the electron in the molecule. An interesting quantity in Huckel theory is the sum of the energies of all the electrons in a molecule, the so called π-electron energy of a molecule.

Prof. Chandrashekara Adiga et al. [7] have defined color energy $E_c(G)$ of a graph G. Motivated by this, we introduced the concept of minimum hub energy E_H (G) of a graph G and computed hub energies of friendship graph, star graph, double star graph, complete graph, and complete bipartite graphs.

A. The minimum hub Energy of a Graph

In the year 2006 the theory of hub numbers introduced by M. Walsh [4]. A hub set in a graph G is a set H of vertices in G such that any two vertices outside H are connected by a path whose internal vertices lie in H. (This includes the degenerate cases where the path consists of the single edge xy or a single vertex x if x=y, call such an H-path trivial). A set H ∈ V (G) is a hub set of G if it has the property that, for any x, y ∈ V (G) − H, there is an H-path in G between x and y. The smallest size of a hub set in G is called the hub number of G, and is denoted by h(G) [6]. For more details on the hub number see [5]. Let G be a simple graph of order n with vertex set V = {v_1, v_2, v_3, ..., v_n } and edge set E. Let H be the minimum hub set of a graph G. The minimum hub matrix of G is the n× n matrix defined by A_H (G) = (a_{ij}), where the entry becomes 1if v_i v_j are adjacent, 1 if i = j and , v_i ∈H, and 0 otherwise.

The characteristic polynomial of A_H (G) is denoted by $f_n(G, \lambda) = \det(\lambda I - A_H(G))$. The minimum hub eigenvalues of the graph G are the eigenvalues of A_H (G).

Since A_H (G) is real and symmetric, its eigenvalues are real numbers and are labelled in non-increasing order $\lambda_1 \geq \lambda_2 \geq \lambda_3 .... \geq \lambda_n$

The minimum hub energy of G is defined as

$$E(G) = \sum_{i=0}^{n} (\lambda_i)$$

B. Minimum Hub Energy of Some Standard Graphs

Definition 2.1 The friendship graph, denoted by $F_3^{(n)}$, is the graph obtained by taking n copies of the cycle graph C_3 with a vertex in common.

Theorem 2.2 If $F_3^{(n)}$ is a friendship graph, then $E_H(F_3^{(n)}) = (2n-1) + 2\sqrt{2n}$

Proof: Let $F_3^{(n)}$ be a friendship graph with $V(F_3^{(n)}) = \{v_0, v_1, v_2, \ldots, v_n\}$.
The minimum hub set = H = $\{v_3\}$. Then Minimum hub matrix is given by

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & \ldots & \ldots & 0 & 0 \\ 1 & 0 & 1 & 0 & \ldots & \ldots & 0 & 0 \\ 1 & 1 & 1 & 1 & \ldots & \ldots & 1 & 1 \\ 0 & 0 & 1 & 0 & \ldots & \ldots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ldots & \ldots & 0 & 1 \\ 0 & 0 & 1 & 0 & \ldots & \ldots & 1 & 0 \end{pmatrix}$$

Characteristic equation is

$$(\lambda - 1)^{n-1}(\lambda + 1)^n[\lambda^2 - 2\lambda - (2n - 1)]$$

Minimum hub Spectra is …

$$\text{spec}_H(F_3^{(n)}) =$$

1(n-1 times), -1(n times), $1+\sqrt{2n}$(one time), $1-\sqrt{2n}$ (one time)

Thus the minimum hub energy of friendship graph is $(2n-1)+ 2\sqrt{2n}$.

Theorem 2.3 If K_n is the complete graph with n vertices has $E_H(K_n) = 2(n - 2)$.

Proof: Let K_n be the complete graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$. The minimum hub number = $h(K_n)$ = 0. The minimum hub matrix is given by

$$\begin{pmatrix} 0 & 1 & 1 & 1 & \ldots & \ldots & 1 & 1 \\ 1 & 0 & 1 & 1 & \ldots & \ldots & 1 & 1 \\ 1 & 1 & 0 & 1 & \ldots & \ldots & 1 & 1 \\ 1 & 1 & 1 & 0 & \ldots & \ldots & 1 & 1 \\ 1 & 1 & 1 & 1 & \ldots & \ldots & 0 & 1 \\ 1 & 1 & 1 & 1 & \ldots & \ldots & 1 & 0 \end{pmatrix}$$

Characteristic polynomial is $(\lambda - (n - 1))(\lambda + 1)^{(n-1)}$
Minimum hub Spectra is n-1 (one time), -1(n-1) time.
Minimum hub energy for complete graph is

$E_H(K_n) = |n - 1| + |- 1|(n - 1)$
$E_H(K_n) = 2(n - 2)$

Theorem 2.4 If $K_{1,n-1}$ is a star graph of order n, then $E_H(K_{1,n-1}) = (4n - 3)$ for $n \geq 2$.
Proof: Let $K_{1,n-1}$ be a graph on n vertices. Minimum hub set is $H = \{v_0\}$.
Then we have the minimum hub matrix as:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & \ldots & \ldots & 1 & 1 \\ 1 & 0 & 0 & 0 & \ldots & \ldots & 0 & 0 \\ 1 & 0 & 0 & 0 & \ldots & \ldots & 0 & 0 \\ 1 & 0 & 0 & 0 & \ldots & \ldots & 0 & 0 \\ 1 & 0 & 0 & 0 & \ldots & \ldots & 0 & 0 \\ 1 & 0 & 0 & 0 & \ldots & \ldots & 0 & 0 \end{pmatrix}$$

$\therefore E_H(K_{1,n-1}) = (4n - 3)$
Theorem 2.6. If $S_{n,n}$ is a double star graph of order $n \geq 3$, then
$E_H(S_{n,n}) = 2[\sqrt{\boldsymbol{n}} + \sqrt{\boldsymbol{n-1}}]$
Proof. Let $S_{n,n}$ be a graph on n vertices. Minimum hub set is $H = \{u_0, v_0\}$.
The proof is similar to the above results. Hence we omit the proof.

Theorem 2.7. The minimum hub energy of the complete bipartite graph is $n+1+\sqrt{n^2 + 2n - 3}$

Proof: Let $K_{n,n}$ be a graph on n vertices. Minimum hub set is $H = \{u_0, v_0\}$.
The proof is similar to the above results. Hence we omit the proof.

C. Properties of minimum Hub Energy of a Graph

Theorem 3.1 Let $|\lambda I - A_H| = a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_n$be the charecterastic polynomial of A_H . then
(i) $a_0 = 1$,
(ii) $a_1 = -|H|$,
(iii) $a_2 = |H|C_2 - |E|$,

Proof. (i) It follows from the definition, $P_H(G, \lambda) := \det(\lambda I - A_H(G))$, that $a_0 = 1$.

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(ii) The sum of the determinants of all of all 1 x 1 principal submatrices of $[A_H]$ is

$$\Rightarrow a_1 = (-1) \text{ trace } [A_H] = -|E|,$$

(iii) The sum of determinants of all the 2 × 2 principal submatrices of $[A_H]$ is

$$= \quad (|H|C_2) - |E|$$

Theorem 3.2. If $\lambda_1, \lambda_2, \ldots, \lambda_n$. are eigenvalues of A_H (G),

then $\lambda_i = |H|$ and $\lambda_i^2 = 2|E| + |H|$.

Theorem 3.3. Let G be a graph with a minimum hub set H. If the minimum hub energy E_H (G) is a rational number, then $E_H (G) \equiv |H| (\text{mod} 2)$.

Proof. Proof is similar to Theorem 3.7 of [1].

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